Journal of Statistical Physics, Vol. 82, Nos. 1/2, 1996

On the Multifractal Analysis of Bernoulli Convolutions. II. Dimensions

François Ledrappier[1] and Anna Porzio[2]

Received January 20, 1995; final May 22, 1995

We show how the formalism developed in a previous paper allows us to exhibit the multifractal nature of the infinitely convolved Bernoulli measures ν_γ for γ the golden mean. In this second part we show how the Hausdorff dimension of the set where the measure has a power law singularity of strength α is related to the large-deviation function given in Part I.

KEY WORDS: Random matrices; thermodynamic formalism; Hausdorff dimension.

1. INTRODUCTION

1.1. The Result

Ws show in this Part II how the formalism of our previous papers can be applied to the multifractal analysis of the infinitely convolved Bernoulli measure associated with the golden number γ. We state that the Hausdorff dimension $f(\alpha)$ of the set where the measure has a power-law singularity of strength α can be computed from the large-deviation function $f(\alpha, l)$ of Part I.

Note that $f(\alpha)$, while obtained as a section of a joint large-deviation function $f(\alpha, l)$,· is in itself intrinsic to the dynamical system (Ω, f, μ).

[1] Laboratoire de Probabilités, Université de Paris VI, 75005 Paris, and École Polytechnique, CMAT, 91128 Palaiseau, France (CNRS URA 169). E-mail: Ledrappier@orphee.polytechnique.fr.

[2] Université Paris XIII, 93430 Villetaneuse, and CPTH École Polytechnique, 91128 Palaiseau, France. E-mail: Porzio@orphee.polytechnique.fr.

0022-4715/96/0100-0397$09.50/0

Indeed, if the pointwise limit

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\log \mu(I(x))}{\log |I(x)|} = \alpha$$

exists and is equal to α on a set B_α of points x, then the limit exists and it is the same for all subsequences of $I(x)$, $x \in B_\alpha$, whose diameter goes to zero. We can then associate with B_α (via the construction of a Frostmann measure associated with G) its Hausdorff dimension $f(\alpha)$.

In ref. 13 we investigated the transformation of the square

$$F(x, y) = \begin{cases} \dfrac{x}{\gamma}, 2y & \text{if} \quad x \leqslant \gamma, \quad y \leqslant 1/2 \\ \dfrac{x}{\gamma} - \gamma, 2y - 1 & \text{if} \quad x \geqslant 1 - \gamma, \quad y \geqslant 1/2 \end{cases}$$

ν_γ is the image of the Sinai–Ruelle–Bowen measure μ via the projection p on the x axis. Clearly F is an endomorphism of the square which possesses two dilating directions. There are very few examples where the mathematics of the multifractal spectrum is well understood: our model is perhaps the first for which it has been possible to obtain a result on multifractal analysis of an invariant measure in the presence of two interacting dilating directions.

In ref. 13 we studied the relations between a Markov partition P_0 for F and the γ-adic partition of the x axis, to establish a dimension formula for the measure ν_γ. The measure ν_γ of a γ-adic interval is computed by counting the rectangles of the Markov partition projecting on it. The dimension of the measure is therefore associated with the growth of a random product of Markov matrices. These matrices are

$$M(n) = \begin{cases} \begin{pmatrix} 1 & k+1 \\ 1 & k+1 \end{pmatrix} & \text{if} \quad n = 2k+2 \\ \begin{pmatrix} 1 & k \\ 1 & k+1 \end{pmatrix} & \text{if} \quad n = 2k+1 \end{cases}$$

and $M(n) \equiv M(x_n)$, where $x_n \in F^{-n} P_0$. If $n_1 \cdots n_q$ is the coding of a γ-adic intervals, then its ν_γ-measure equals

$$\frac{|M(x_{n_q}) \cdots M(x_{n_1})|}{2^{(x_{n_1} + \cdots + x_{n_q})}}$$

and its length equals $l(x_{n_1}) + \cdots + l(x_{n_q}))$ (see ref. 13 for more details).

We denoted by δ the almost sure value of the local dimension. We take the notations of Sections 9 and 10 of Part I. Recall that $f<0$ and $\alpha_2<0$. Recall that the function $\beta \to F(\beta)$ and its Legendre transform $\alpha \to f(\alpha+\delta)$ are defined for β, and then for α, near to zero (cf. Part I). Let

$$S(\alpha+\delta)=\left\{x \text{ such that } \lim_{n\to\infty} \frac{\log[\,|M(x_n)\cdots M(x_0)u|/2^{(x_0+\cdots+x_n)}}{(l(x_0)+\cdots+l(x_n))}=\alpha+\delta\right\}$$

and let $pS(\alpha+\delta)$ be its projection on $[0, 1]$. In this paper we prove the following result.

Theorem. For $|\alpha|$ sufficiently small,

$$HD(pS(\alpha+\delta)) = -f(\alpha+\delta)+(\alpha+\delta)$$

We prove first an upper bound to the Hausdorff dimension. This is done by an easy covering argument. The opposite inequality is more difficult and requires the construction of a Frostmann measure.

1.2. Constructing the Frostmann Measure m^*

We shall construct the unique "Gibbs measure" $m \equiv m_{\beta,F}$ associated with the "pressure" G, and depending also on β and F. For a suitable choice of the parameters, $m_{\beta,F}$ will be supported on the set of trajectories such that

$$\frac{\log[\,|M(x_n)\cdots M(x_0)u|/2^{(x_0+\cdots+x_n)}]}{(l(x_0)+\cdots+l(x_n))} \sim \alpha+\delta$$

The results of Section 10, Part I, show that a good choice is $\beta=\beta^*$ and $F=F^*$ corresponding to the value $\alpha_2(c)$ which maximizes $\sigma((\alpha+\delta)\alpha_2, \alpha_2)/\alpha_2$.

In fact we shall consider $m_{\beta,F}$ as invariant measures for the shift on the space of trajectories $x_0, x_1,..., x_i \in N$, and will interpret the large-deviation theorem of Section 10, Part I, as an entropy/dimension formula. More precisely, recall the Ledrappier–Young formula[14] in two dimensions, that is,

$$h_2(m_{\beta^*,F^*}) - h_1(m_{\beta^*,F^*}) = \lambda_2(m_{\beta^*,F^*})(\delta_2(m_{\beta^*,F^*}) - \delta_1(m_{\beta^*,F^*}))$$

where h, λ, and δ are entropy, exponent, and dimension, respectively. Here $h_2-h_1=[f(\alpha+\delta)-(\alpha+\delta)]\,\alpha_2(c)$, $\lambda_2=\alpha_2(c)$, and $\delta_2-\delta_1=-f(\alpha+\delta)-(\alpha+\delta)$. It follows that $-f(\alpha+\delta)+\alpha+\delta$ is the dimension of the projected measure of m_{β^*,F^*}.

We will construct in this way a measure $m_{\beta^*, F^*} \equiv m^*$ whose projection ν^* is the Frostmann measure: the projection on $[0, 1]$ of the set of trajectories such that

$$\frac{\log[|M(x_n) \cdots M(x_0)u|/(g(x_0) + \cdots + g(x_n))]}{\log(l(x_0) + \cdots + l(x_n))} \sim \alpha + \delta$$

(the support of ν^*) has dimension $-f(\alpha+\delta) + (\alpha+\delta)$.

This program, simple in its main lines, requires considerable technical work. We will construct m^* as limit of a sequence of approximated measures m_n: this construction is inspired by the classical construction of the Gibbs measures. Moreover, as m^* is not *a priori* a nice measure (e.g., it might be noninvariant), we will introduce an auxiliary equivalent measure with good ergodic properties to work with. Courage!

2. UPPER BOUND

Let

$$S(\alpha+\delta) = \left\{ \text{x such that } \lim_{n \to \infty} \frac{\log[|M(x_n) \cdots M(x_0)u|/2^{(x_0) + \cdots + x_n)}]}{(l(x_0) + \cdots + l(x_n))} = \alpha + \delta \right\}$$

and let $pS(\alpha+\delta)$ be its projection on $[0, 1]$. We can state the following result.

Theorem 2.1. For $|\alpha|$ sufficiently small

$$HD(pS(\alpha+\delta)) = -f(\alpha+\delta) + (\alpha+\delta)$$

We start by proving an upper bound to the Hausdorff dimension. Recall that the function $\beta \to F(\beta)$ and its Legendre transform $\alpha \to f(\alpha+\delta)$ are defined for β, and then for α, near zero (cf. Part I).

Lemma 2.2. *Upper Bound.* For $|\alpha|$ sufficiently small

$$HD(pS(\alpha+\delta)) \leqslant -f(\alpha+\delta) + (\alpha+\delta)$$

Proof. Let

$$S_n(\alpha+\delta, \alpha_2, \varepsilon')$$

$$= \left\{ x_0 \cdots x_n \text{ such that } \log \frac{|M(c_n) \cdots M(x_0)u|}{2^{(x_0 + \cdots + x_n)}} \in ((\alpha+\delta)\alpha_2 - \varepsilon', (\alpha+\delta)\alpha_2 + \varepsilon') \right.$$

$$\left. \text{and } (l(x_0) + \cdots + l(x_n)) \in (\alpha_2 - \varepsilon', \alpha_2 + \varepsilon') \right\}$$

and $\alpha_2 \in Q(k, m) = \{k2^{-m}\}_{k \in Z, m \in N}$. Let $p(S_n(\alpha+\delta, \alpha_2, \varepsilon')$ be its projection on the x axis.

We consider the ρ-Hausdorff measure of $pS_n(\alpha+\delta, \alpha_2, \varepsilon')$. We have first

$$HDM_{\rho,\varepsilon}(pS_n(\alpha+\delta, \alpha_2, \varepsilon'))$$

$$= \inf_{\{\text{coverings } U_1 \text{ of size } 2^{n\alpha_2} < \varepsilon\}} \sum |U_i|^\rho$$

$$\leqslant 2^{n(\alpha_2+\varepsilon')\rho} 2^{n\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon')} 2^{-n(\alpha_2+\varepsilon')(\alpha+\delta)}$$

because we can cover $pS_n(\alpha+\delta, \alpha_2, \varepsilon')$ with

$$2^{n\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon')} 2^{-n(\alpha_2+\varepsilon')(\alpha+\delta)}$$

intervals. This follows at once because we can cover $S_n(\alpha+\delta, \alpha_2, \varepsilon')$ with

$$2^{n\sigma((\alpha+\beta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon')} 2^{n \log_\beta 2(\alpha_2+\varepsilon')}$$

intervals of length $2^{n(\alpha_2+\varepsilon')}$, then dividing by the ambiguity, which is equal to

$$2^{n \log_\beta 2(\alpha_2+\varepsilon')} 2^{n(\alpha_2+\varepsilon')(\alpha+\delta)}$$

we have the result. The signs + or − are to be chosen accordingly with σ.

If

$$\rho > \frac{-\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon') + (\alpha_2 \pm \varepsilon')(\alpha+\delta)}{\alpha_2 \pm \varepsilon'}$$

as ε' has been chosen small, by letting ε be sufficiently small we have, uniformly in n, $HDM_\rho(pS_n(\alpha+\delta, \alpha_2, \varepsilon')) \leqslant 1$.

It follows that

$$HDM_\rho(\liminf_n pS_n(\alpha+\delta, \alpha_2, \varepsilon'))$$

$$\leqslant \liminf_n HDM_\rho(pS_n(\alpha+\delta, \alpha_2, \varepsilon')) \leqslant 1$$

if

$$\rho > \frac{-\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon') + (\alpha_2 \pm \varepsilon')(\alpha+\delta)}{\alpha_2 \pm \varepsilon'}$$

And so we have the Hausdorff dimension (HD):

$$HD(\liminf_n pS_n(\alpha+\delta, \alpha_2, \varepsilon') \leqslant \frac{-\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon') + (\alpha_2 \pm \varepsilon')(\alpha+\delta)}{\alpha_2 \pm \varepsilon'}$$

Now, we have that

$$pS(\alpha+\delta) \subset \bigcup_{\alpha_2 \in Q(k,n)} \liminf_n pS_n(\alpha+\delta, \alpha_2, \varepsilon')$$

which implies that

$$\begin{aligned} HD(pS(\alpha+\delta)) &\leqslant \sup_{\alpha_2 \in Q(k,n)} HD(\liminf_n pS_n(\alpha+\delta, \alpha_2, \varepsilon')) \\ &= \sup_{\alpha_2 \in Q(k,n)} \frac{-\sigma((\alpha+\delta)\alpha_2 \pm \varepsilon', \alpha_2 \pm \varepsilon') + (\alpha_2 \pm \varepsilon')(\alpha+\delta)}{\alpha_2 \pm \varepsilon'} \\ &\leqslant \sup_{\alpha_2 \in R} \frac{-\sigma((\alpha+\delta)\alpha_2, \alpha_2) + \alpha_2(\alpha+\delta)}{\alpha_2} + \varepsilon'' \end{aligned}$$

We recover the equation for α_2 which we studied in Section 10 of Part I. Its solution gives $\alpha_2 = \alpha_2(c)$, which is the point such that $\sigma((\alpha+\delta)\alpha_2, \alpha_2) = \alpha_2 f(\alpha+\delta)$.

In conclusion, we have shown that $HD(pS(\alpha+\delta)) \leqslant -f(\alpha+\delta) + (\alpha+\delta)$.

Lemma 2.3. *Lower Bound.* For $|\alpha|$ sufficiently small

$$HD(pS(\alpha+\delta)) \geqslant -f(\alpha+\delta) + (\alpha+\delta)$$

To prove this lemma, we shall construct a measure m^* and its projection ν^* ("Frostmann measure") which will be supported on the set $pS((\alpha+\delta), \alpha_2(c))$, and whose dimensions is $-f(\alpha+\delta) + (\alpha+\delta)$. We have

$$\begin{aligned} HD(\operatorname{supp} \nu^*) &= HDpS((\alpha+\delta), \alpha_2(x)) \\ &\geqslant HD \text{ measure} \\ &= -f(\alpha+\delta) + (\alpha+\delta) \end{aligned}$$

Similarly,

$$pS(\alpha+\delta) = \bigcup_{\alpha_2} pS((\alpha+\delta)\, \alpha_2 \supset pS((\alpha+\delta), \alpha_2(c))$$

so that

$$HD(pS(\alpha+\delta) \geqslant HDpS((\alpha+\delta), \alpha_2(c)) \geqslant HDv^* = -f(\alpha+\delta)+(\alpha+\delta)$$

3. STEPS OF THE CONSTRUCTION OF v*

3.1. The "Approximate" Measure m_n

Choose β and F near zero. Consider the sequence m_n of approximated measures, defined on X and supported on the set of the sequences of $X = N^N$, $\mathbf{x} = x_0 x_1 \cdots x_i \cdots$, which coincide for $i > n$, i.e., such that $x_{i>n} \equiv \hat{x}_{i>n}$, where $\hat{x}_{i>n}$ is a arbitrarily chosen sequence (e.g., the sequence 0, 0, 0,...):

$$\begin{aligned} m_n(x_0 x_1 \cdots x_n) = &\left(\frac{M(x_0)\, M(x_1) \cdots M(x_n)(v_{n+1})}{2^{x_0 + x_1 \cdots + x_n}}\right)^\beta \\ &\times \gamma^{F(x_0 + x_1 + \cdots + x_n)} \pi(x_0 x_1 \cdots x_n)\, \delta(x_0, ..., x_n, \mathbf{0}) \\ &\times \left\{ \sum_{x_1 \cdots x_n} \left(\frac{M(x_0)\, M(x_1) \cdots M(x_n)(v_{n+1})}{2^{x_0 + x_1 \cdots + x_n}}\right)^\beta \right. \\ &\left. \times \gamma^{F(x_0 + x_1 + \cdots + x_n)} \pi(x_n x_1 \cdots x_n) \right\}^{-1} \end{aligned}$$

and

$$m_n(x_0 \cdots x_{n+1}) = \begin{cases} 0 & \text{if } \; x_{n+1} \neq 0 \\ m_n(x_0 \cdots x_n) & \text{otherwise} \end{cases}$$

3.2. Convergence of m_n

We shall prove the existence of the limit measure m^* by an argument which is usual in Gibbs measure theory, that is, the convergence of conditional measures of m_n. The contraction properties of matrices M (Part I) allows us to show explicitly this convergence.

Since the configuration space X is not compact, we will also need a property which state that the measures m_n are completely determined by the values they take on compact subsets of X.

Recall that a family of measures μ_n is "tight" if $\forall \varepsilon > 0$ it is possible to find a compact set X_ε each such $\mu(X/X_\varepsilon) > 1 - \varepsilon$ for all n, and that a tight family of probability measures on a locally compact and separated space is relatively compact.[15]

Lemma 3.1. For $|\beta|$ and F sufficiently small the family $\{m_n\}$ is tight.

Proof. Choose $X_\varepsilon = \{\text{sequences } \boldsymbol{\sigma} \text{ s.t. } \forall i\ \sigma_i \leqslant p\}$, where p is an integer (huge). We check that uniformly in n

$$m_n(\{\mathbf{x}: x_i > p\ \forall i = 1,..., n\}) < \varepsilon$$

We have, if β is positive,

$$\begin{aligned}
&m_n(\{\mathbf{x}: x_i > p\}) \\
&\leqslant C_n \sum_{x_0 \cdots x_n,\, x_i > p} \left(\frac{|S_n|}{2^{x_0 + \cdots x_n}}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{(x_0 + \cdots x_n)F} \pi_n \\
&\leqslant D_n \sum_{x_0 \cdots x_n,\, x_i > p} \left(\frac{x_0 \cdots x_n}{2^{x_0 + \cdots x_n}}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{(x_0 + \cdots x_n)F} 2^{(x_0 + \cdots x_n)} \\
&\leqslant D_n \sum_{x_0 > p} \left(\frac{x_0}{2^{x_0}}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{Fx_0} 2^{-x_0} \\
&\quad \times \sum_{x_1 > p} \left(\frac{x_1}{2^{x_1}}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{Fx_1} 2^{-x_1} \cdots \\
&\quad \times \cdots \sum_{x_n > p} \left(\frac{x_n}{2^{x_n}}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{Fx_n} 2^{-x_n} \\
&\leqslant E_n \left(\sup_{k > p} \left(\frac{k}{2^k}\right)^\beta \left(\frac{1-\sqrt{5}}{2}\right)^{kF} 2^{-k}\right)^n
\end{aligned}$$

where C_n is normalization constant (smaller than 2^n), D_n comes from the "factorization" of the product of matrices and is smaller than $2^n C_n$, and E_n a term bounded by E^n for some E.

If $\beta < 0$ and if β and F are small, we similarly have

$$m_n(X_\varepsilon) < A^{pn} \qquad \text{for} \quad A < 1$$

Then $\forall \varepsilon$ positive, $\exists p \equiv p(\varepsilon)$ such that uniformly in n we have $m_n(X_\varepsilon) < \varepsilon$.

Remark. The family $\{m_n(\beta, F)\}$ is actually tight for $\beta \in (-1, \infty)$.

Since the m_n are a tight family of probability measures on X locally compact and separated, there exists a subsequence m_{n_s} of measures weakly converging to some limit probability m: $\int f\, dm_{n_s} \to \int f\, dm$ $\forall$ continuous function f.[15]

We now show the convergence of the conditional measures of m_n. This provides the relation that any limit measure μ must satisfy (in terms of its explicit expression via its conditional measures). Moreover, we show that a measure m which satisfies such a relation is necessarily unique. It follows that any converging subsequence m_{n_s} converges to the same limit; therefore the whole sequence does converge. Here we go!

Let

$$C=\begin{pmatrix} 0 & \cdots & k \\ x_0 & \cdots & x_k \end{pmatrix}$$

a cylinder.

Let us first show that

$$\exists \lim_{N\to\infty} m_N(x_0\cdots x_k \mid x_{k+1}\cdots x_N)=m(x_0\cdots x_k \mid x_{k+1}\cdots)$$

We have

$$\begin{aligned} &m_N(x_0\cdots x_k \mid x_{k+1}\cdots x_N) \\ &\quad=\frac{m_N(x_0\cdots x_k x_{k+1}\cdots x_N)}{\sum_{x'_0\cdots x'_k} m_N(x'_0\cdots x'_k x_{k+1}\cdots x_N)} \\ &\quad=|M(x_0)\,M(x_1)\cdots M(x_k)\cdots M(x_N)\,u_{N+1}|^{\beta}\pi(x_0\cdots x_n)\gamma^{F(x_0+\cdots+x_n)} \\ &\qquad\times\left\{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\,M(x_{k+1})\cdots M(x_N)\,u_{N+1}|^{\beta}\right. \\ &\qquad\left.\times\,\pi(x'_0\cdots x'_n)\gamma^{F(x'_0+\cdots+x'_n)}\right\}^{-1} \end{aligned}$$

where we define $u_{N+1}=\lim_{M\to\infty} M(x_{N+1})\cdots M(x_{N+M})u$, u being any vector of S. This definition is legitimate, because we know (Part I) that $\delta(S_N u, S_N v)\leqslant \rho^N\delta(u, v)$ with $\rho<1$ if $M\neq\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and otherwise, if S_N is the product of N matrices $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and u, v, are, respectively, equal to $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ (this being the case where the least contraction atcs on the farthest vectors of the support), then $S_N=\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right)$ and

$$\delta(S_N u, S_N v)=\delta\left(\begin{pmatrix}1\\N+1\end{pmatrix},\begin{pmatrix}1\\N\end{pmatrix}\right)\to 0 \qquad \text{when} \quad N\to\infty$$

Then the limit which defines u_{N+1} exists independently of u, but it depends, continuously, on the sequence x_i for $i>N$.

Then, $\lim_{N\to\infty} M(x_{k+1})\cdots M(x_N)\,u_{N+1}\to u_{k+1}$ and $m_N(x_0\cdots x_k \mid x_{k+1}\cdots x_N)$ converges uniformly (in $x_{K+1}\cdots$) toward its limit $m(x_0\cdots x_k \mid x_{k+1}\cdots)$, which gives the expression for the conditional measures of any weak limit m as above:

$$\begin{aligned}\lim_{N\to\infty} & m_N(x_0\cdots x_k \mid x_{k+1}\cdots x_N)\\ &= \frac{|M(x_0)\,M(x_1)\cdots M(x_k)\,u_{k+1}|^\beta \pi(x_0\cdots x_n)\,\gamma^{F(x_0+\cdots+x_n)}}{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\,u_{k+1}|^\beta \pi(x'_0\cdots x'_n)\,\gamma^{F(x'_0+\cdots+x'_n)}}\\ &\equiv m(x_0\cdots x_k \mid x_{k+1}\cdots)\end{aligned}$$

Therefore $m(x_0\cdots x_k \mid x_{k+1}\cdots)$ is a continuous function of $\mathbf{x}$. Since the family m_N is tight, we have in particular that

$$\lim_{s\to\infty}\int m_{N_s}(x_0\cdots x_k \mid \hat{x}_{k+1}\cdots \hat{x}_{N_s})\,dm_{N_s}(\hat{\mathbf{x}})$$

(because of the uniform convergence)

$$\int \lim_{s\to\infty}\,[m_{N_s}(x_0\cdots x_k \mid \hat{x}_{k+1}\cdots\hat{x}_{N_s}) - m(x_0\cdots x_k \mid \hat{x}_{k+1}\cdots)]\,dm_{N_s}(\hat{\mathbf{x}})$$

$$+\lim_{s\to\infty}\int m(x_0\cdots x_k \mid \hat{x}_{k+1}\cdots)\,dm_{N_s}(\hat{\mathbf{x}})$$

$$=\int m(x_0\cdots x_k \mid \hat{x}_{k+1}\cdots)\,dm(\hat{\mathbf{x}}) \equiv m(x_0\cdots x_k) = m(C)$$

where C is the cylinder

$$C = \left(\frac{0\ \cdots\ k}{x_0\cdots x_k}\right)$$

3.3. Uniqueness

Lemma 3.2. There exist a real, positive constant c such that

$$c^{-1} \leqslant \frac{m(x_0\cdots x_k \mid \tilde{x}_{k+1}\cdots)}{m(x_0\cdots x_k \mid \tilde{\tilde{x}}_{k+1}\cdots)} \leqslant c$$

Proof. We have

$$\begin{aligned}&\frac{m(x_0\cdots x_k \mid \tilde{x}_{k+1}\cdots)}{m(x_0\cdots x_k \mid \tilde{\tilde{x}}_{k+1}\cdots)}\\ &= \left\{\frac{|M(x_0)\,M(x_1)\cdots M(x_k)\tilde{u}_{k+1}|^\beta \pi(x_0\cdots x_n)\gamma^{F(x_0+\cdots+x_n)}}{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\tilde{u}_{k+1}|^\beta \pi(x'_0\cdots x'_n)\gamma^{F(x'_0+\cdots+x'_n)}}\right\}\\ &\quad\times\left\{\frac{|M(x_0)\,M(x_1)\cdots M(x_k)\tilde{\tilde{u}}_{k+1}|^\beta \pi(x_0\cdots x_n)\gamma^{F(x_0+\cdots+x_n)}}{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\tilde{\tilde{u}}k+1|^\beta \pi(x'_0\cdots x'_n)\gamma^{F(x'_0+\cdots+x'_n)}}\right\}^{-1}\end{aligned}$$

where, as above, $\tilde{u}_{k+1} = \lim_{N\to\infty} M(\tilde{x}_{k+1})\cdots M(\tilde{x}_N)u$.

We study first the quotient

$$\frac{|M(x_0)\,M(x_1)\cdots M(x_k)\tilde{u}_{k+1}|^{\beta}\pi(x_0\cdots x_n)\gamma^{F(x_0+\cdots+x_n)}}{|M(x_0)\,M(x_1)\cdots M(x_k)\tilde{u}_{k+1}|^{\beta}\pi(x_0\cdots x_n)\gamma^{F(x_0+\cdots+x_n)}}$$

$$\equiv\frac{|M(x_0)\,M(x_1)\cdots M(x_{k)}\,{}_{\tilde{u}}^{k+1}|^{\beta}}{|M(x_0)\,M(x_1)\cdots M(x_k)\,\tilde{\tilde{u}}_{k+1}|^{\beta}}$$

Let $\tilde{u}_0=M(x_0)\cdots M(x_k\,\tilde{u}_{k+1}$.

This quotient can be bounded in terms of the distance between the vectors $\tilde{u}_0$ and $\tilde{u}\in S[\pi/4,\pi/2]$: $|\tilde{u}_0|/|\tilde{\tilde{u}}_0|=1+\gamma\delta(\tilde{u}_0,\tilde{\tilde{u}}_0)$, where γ is a constant (cf. Part I). Therefore this quotient can be bounded above by 2 and below by 1/2.

Similarly, since

$$\inf_j\left(\frac{a_j}{b_j}\right)\leqslant\frac{\sum_{k=0}^{n}a_k}{\sum_{k=0}^{n}b_k}\leqslant\sup_j\left(\frac{a_j}{b_j}\right)$$

we have that the quotient R,

$$R=\frac{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\,\tilde{u}_{k+1}|^{\beta}\pi(x'_0\cdots x'_n)\gamma^{F(x'_0+\cdots+x'_n)}}{\sum_{x'_0\cdots x'_k}|M(x'_0)\,M(x'_1)\cdots M(x'_k)\,\tilde{\tilde{u}}k+1|^{\beta}\pi(x'_0\cdots x'_n)\gamma^{F(x'_0+\cdots+x'_n)}}$$

is bounded above and below by

$$\inf_k\inf_{\tilde{u}_{k+1},\tilde{\tilde{u}}_{k+1}}\frac{|M(x_0)\,M(x_1)\cdots M(x_k)\,\tilde{\tilde{u}}_{k+1}|^{\beta}}{|M(x_0)\,M(x_1)\cdots M(x_k)\,\tilde{\tilde{u}}_{k+1}|^{\beta}}$$

$$\leqslant R\leqslant\sup_k\sup_{\tilde{u}_{k+1},\tilde{\tilde{u}}_{k+1}}\frac{|M(x_0)\,M(x_1)\cdots M(x_k)\,\tilde{\tilde{u}}_{k+1}|^{\beta}}{|M(x_0)\,M(x_1)\cdots M(x_k)\,\tilde{\tilde{u}}_{k+1}|^{\beta}}$$

Therefore we can choose $c=4$ in the lemma.

Let us show the uniqueness.

Proposition 3.3. Let $m(x_0\cdots x_k\mid x_{k+1}\cdots)$ be a family of functions satisfying

$$c^{-1}\leqslant\frac{m(x_0\cdots x_k\mid\tilde{x}_{k+1}\cdots)}{m(x_0\cdots x_k\mid\tilde{\tilde{x}}_{k+1}\cdots)}\leqslant c$$

uniformly in k. Then there exists at most one measure m such that $m(x_0\cdots x_k\mid x_{k+1}\cdots)$ is the family of conditional measures of m.

Proof. Let m and m_1 be two probability measures with conditional measures $m(x_0 \cdots x_k \mid x_{k+1} \cdots)$. We repeat the classical argument[17] to show that m is absolutely continuous with respect to m_1 with Radon-Nikodym derivative $h(\mathbf{x})$ bounded above and below by a constant. Then, since m and m_1 have the same conditional measures, $h(\mathbf{x})$ depends only on conditioning, and finally $h(\mathbf{x})$ is a constant equal to 1.

We can write, $\forall$ cylinder

$$C = \begin{pmatrix} 0 & \cdots & k \\ x_0 & \cdots & x_k \end{pmatrix}$$

that

$$\begin{aligned}
&m(x_0 \cdots x_k) \\
&\quad = \int_{\tilde{\mathbf{x}}} m(x_0 \cdots x_k)\, m_1(\tilde{\mathbf{x}}) \\
&\quad = \int_{\tilde{\mathbf{x}}} m_1(\tilde{\mathbf{x}}) \int_{\hat{\mathbf{x}}} m(x_0 \cdots x_k \mid \hat{x}_{k+1} \cdots)\, m(\hat{\mathbf{x}}) \\
&\quad = \int_{\tilde{\mathbf{x}}} m_1(\tilde{\mathbf{x}}) \int_{\hat{\mathbf{x}}} m(\tilde{\mathbf{x}}) \frac{m(x_0 \cdots x_k \mid \hat{x}_{k+1} \cdots)}{m(x_0 \cdots x_k \mid \tilde{x}_{k+1} \cdots)}\, m_1(x_0 \cdots x_k \mid \tilde{x}_{k+1} \cdots) \\
&\quad \leqslant c \int_{\tilde{\mathbf{x}}} m_1(\tilde{\mathbf{x}}) \int_{\hat{\mathbf{x}}} m(\hat{\mathbf{x}})\, m_1(x_0 \cdots x_k \mid \tilde{x}_{k+1} \cdots) \\
&\quad = c \int_{\tilde{\mathbf{x}}} m_1(\tilde{\mathbf{x}})\, m_1(x_0 \cdots x_k \mid \tilde{x}_{k+1} \cdots) \int_{\hat{\mathbf{x}}} m(\hat{\mathbf{x}}) \\
&\quad = c m_1(x_0 \cdots x_k)
\end{aligned}$$

Then, for any cylinder

$$C = \begin{pmatrix} 0 & \cdots & k \\ x_0 & \cdots & x_k \end{pmatrix}$$

we have, exchanging the roles of m and m_1,

$$c^{-1} \leqslant \frac{m(x_0 \cdots x_k)}{m_1(x_0 \cdots x_k)} \leqslant c$$

Then for any Borel set, $dm(\mathbf{x}) = h(\mathbf{x})\, dm_1(\mathbf{x})$ with $c^{-1} < h(\mathbf{x}) < c$.

By definition of conditional measure, the density h depends only on coordinates $> k$. If h were not a constant, h^q would be also a density, contradicting that $c^{-1} < h < c$.

3.4. Invariance

We have, ∀ cylinder

$$C=\begin{pmatrix} 0 & \cdots & k \\ x_0 & \cdots & x_k \end{pmatrix}$$

that

$$\begin{aligned} &m(x_0 \cdots x_k) \\ &\quad = \int_{\hat{\mathbf{x}}} m(x_0 \cdots x_k \mid \hat{x}_{k+1} \cdots)\, dm(\hat{\mathbf{x}}) \\ &\quad = \int_{\hat{x}_0 \cdots \hat{x}_k} \Big\{ |M(x_0)\, M(x_1) \cdots M(x_k)\, \hat{u}_{k+1}|^{\beta} \pi(x_0 \cdots x_n) \gamma^{F(x_0 + \cdots + x_n)} \\ &\qquad \times \Big[\sum_{x'_0 \cdots x'_k} |M(x'_0)\, M(x'_1) \cdots M(x'_k)\, u_{k+1}|^{\beta} \\ &\qquad \times \pi(x'_0 \cdots x'_n) \gamma^{F(x'_0 + \cdots + x'_n)} \Big]^{-1} \Big\}\, dm(\hat{x}) \end{aligned}$$

The translated measure $\tau^{p^*} m$ of the same cylinder

$$C=\begin{pmatrix} 0 & \cdots & k \\ x_0 & \cdots & x_k \end{pmatrix}$$

is equal to the measure m of the cylinder

$$\tau^{-p} C=\begin{pmatrix} p & \cdots & k+p \\ x_0 & \cdots & x_k \end{pmatrix}$$

which is

$$\begin{aligned} &\int dm(\hat{x}) \sum_{\hat{x}_{-p} \cdots \hat{x}_0 \hat{x}_1 \cdots \hat{x}_N \cdots} \\ &\quad \times \Big\{ |M(\hat{x}_{-p} \cdots)\, M(\hat{x}_{-1})\, M(x_0) \cdots M(x_k) \hat{u}_{k+1}|^{\beta} \pi(x_0 \cdots x_n) \gamma^{F(x_0 + \cdots + x_n)} \\ &\quad \times \Big[\sum_{x'_0 \cdots x'_k} |M(\hat{x}_{-p}) \cdots M(\hat{x}_{-1})\, M(x'_0) \cdots M(x'_k) \hat{u}_{k+1}|^{\beta} \\ &\quad \times \pi(x'_0 \cdots x'_n) \gamma^{F(x'_0 + \cdots + x'_n)} \Big]^{-1} \Big\} \end{aligned}$$

So we introduce a new measure $\tilde{\mu}$ on Z, by giving its conditional measures:

$$\tilde{\mu}(x_{-k}\cdots x_k \mid \hat{x}_j, j>k, \tilde{x}_j, j<-k)$$
$$=\frac{\langle \tilde{u}_{-(k+1)}, M(x_{-k}\cdots)\cdots M(x_k)\hat{u}_{k+1}\rangle^{\beta}\pi(x_0\cdots \mid x_n)\gamma^{F(x_0+\cdots+x_n)}}{\sum_{x'_k\cdots x'_k}\langle \tilde{u}_{k+1}, M(x'_{-k})\cdots M(x'_k)\,\tilde{u}_{k+1}\rangle^{\beta}\pi(x'_0\cdots x'_n)\gamma^{F(x_0+\cdots+x'_n)}}$$

By the same arguments as in Proposition 3.3, there exists a unique measure $\tilde{\mu}$ possessing the above conditional measures. In particular, since the expression giving these conditional measures is clearly stationary, the measure $\tau\tilde{\mu}$ has the same conditional measures as $\tilde{\mu}$. The measure $\tilde{\mu}$ is therefore τ invariant.

Similarly, we can prove that the restriction of $\tilde{\mu}$ to Z^+, which we denote by $\tilde{\mu}^+$, and which is unique and invariant, is equivalent to m.

We have shown the following result.

Proposition 3.4. For all β, F sufficiently small, there exists a unique invariant measure $\tilde{\mu}^+$ on (Ω^+, τ) absolutely equivalent to $m_{\beta,F}$.

3.5. Ergodicity

As in the classical case, we show the ergodicity of $\tilde{\mu}^+$ by the same arguments which prove its unicity. Indeed, as in the classical case, it is easy to show a stronger property:

Proposition 3.5.[17] The dynamical system $(\Omega, \tau, \tilde{\mu}^+)$ is a K system.

Proof. Let $B(\infty)_{\tilde{\mu}^+} = \bigcap_{n\in N} B(\Lambda_n^C)_{\tilde{\mu}^+}$, where $B(\Lambda_n^C)_{\tilde{\mu}^+}$ is the $\tilde{\mu}^+$-completion of $B(\Lambda_n^C)$, the σ-algebra generated by the cylinders which do not depend on $[-n, n]$. It is sufficient to observe that $B(\infty)_{\tilde{\mu}^+}$ is a trivial σ-algebra, i.e., that any $B(\infty)_{\tilde{\mu}^+}$-measurable function f is almost everywhere constant. Then f is necessarily trivial, and the system is a K system.

3.6. Limit Theorems

Recall that m is the weak limit of the m_n (Section 3.1), which is not *a priori* an invariant measure, and that $\tilde{\mu}^+$ is invariant ergodic and equivalent to m. We shall write the exponent, the entropy, etc., of m_k because only they are explicit, in the form of suitable sums and we show that these sums have the same limit behavior as the ergodic sums of $\tilde{\mu}^+$.

By the ergodic theorem, we have $\tilde{\mu}^+$ almost everywhere the following limits:

$$\lim_{n \to \infty} \frac{1}{n} \log \gamma^{x_0 + \cdots + x_n} \to \lambda_2 \equiv \alpha_2(c)$$

$$\frac{1}{n} (l(x_0) + \cdots l(x_n)) \to \alpha_2$$

$$\frac{1}{n} \log \frac{|M(x_0) \cdots M(x_n)|}{2^{(x_0 + \cdots x_n)}} \to \alpha_2(\alpha + \delta)$$

because the x_i are distributed according to m, that is, according to $\tilde{\mu}^+$, which is ergodic, and these expressions are ergodic sums.

3.7. Exponent

Define the exponent $\lambda_2(k)$ by

$$\lambda_2(k) = \frac{1}{k} \sum_{x_0 \cdots x_k} m_k(x_0 \cdots x_k) \log \gamma^{x_0 + \cdots + x_k}$$

Lemma 3.6. We have

$$\lim_{k \to \infty} \lambda_2(k) = \lambda_2$$

Proof. Consider the sum

$$\frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} x_i \, dm_k$$

Fix an index i and consider the integral

$$\int_\Omega x_i \, dm_k$$

Choose p large enough that, uniformly in k,

$$\int_{x_0 \cdots x_k > p} x_i \, dm_k \leqslant \varepsilon(p), \qquad \int_{x_0 \cdots x_k > p} x_i^2 \, dm_k \leqslant \varepsilon(p)$$

(cf. Lemma 3.1).

We claim that $\int x_i \, dm_k \to \int x_i \, dm$ when $k \to \infty$. Compare the integrals $m(x_i)$ and $m_k(x_i)$. We can decompose, keeping x_i fixed, the measure m_k according to its conditional measures (of x_i given Ω/x_i):

$$\int_{\Omega/x_i} dm_k(\mathbf{x})$$

$$= \int_{\Omega/x_i} dm_k(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)$$

$$= \sum_{x_i} x_i m_k(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)$$

$$= \int_{\Omega/x_i} \frac{dm_k(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)}{dm(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)} dm(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)$$

$$\times \sum_{x_i} x_i \frac{m_k(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)}{m(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)} m(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)$$

Consider first the quotient

$$\frac{m_k(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)}{m(x_0 \cdots x_{i-1} \, |x_i| \, x_{i+1} \cdots x_k \cdots)}$$

or, more explicitly,

$$\Bigg(|M(x_0)\, M(x_1) \cdots M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^{\beta}$$
$$\times \pi(x_0 \cdots | \, x_k) \gamma^{F(x_0 + \cdots + x_k)}$$
$$\times \Bigg\{ \sum_{x_i'} |M(x_0)\, M(x_1) \cdots M(x_{i-1})\, M(x_i')\, M(x_{i+1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^{\beta}$$
$$\times \pi(x_0 \cdots x_k) \gamma^{F(x_0 + \cdots + x_k)} \Bigg\}^{-1} \Bigg)$$
$$\times \Bigg(|M(x_0) \cdots M(x_{i-1})\, M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^{\beta}$$
$$\times \pi(x_0 \cdots x_k) \gamma^{F(x_0 + \cdots + x_k)}$$
$$\times \Bigg\{ \sum_{x_i'} |M(x_0) \cdots M(x_{i-1})\, M(x_i')\, M(x_{i-1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^{\beta}$$
$$\times \pi(x_0 \cdots x_k) \gamma^{F(x_0 + \cdots + x_k)} \Bigg\}^{-1} \Bigg)^{-1}$$

Consider now the quotient of the numerators:

$$|M(x_0)\, M(x_1) \cdots M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^{\beta} \pi(x_0 \cdots x_k) \gamma^{F(x_0 + \cdots + x_k)}$$
$$\times [\, |M(x_0) \cdots M(x_{i-1})\, M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \hat{u}_{k+1}|^{\beta}$$
$$\times \pi(x_0 \cdots x_k) \gamma^{Fx_0 + \cdots + x_k)}]^{-1} \equiv A$$

If $k-i$ is large, we have the quotient of two vectors which which are very near, and as $|v_1|/|v_2| \sim 1 + c \times \delta(v_1, v_2)$, we can bound A above and below:

$$\begin{aligned} 1-\frac{c}{k-i} \sim 1-c\delta\left(\begin{pmatrix}1\\ k-i\end{pmatrix}, \begin{pmatrix}1\\ k-i+1\end{pmatrix}\right) \\ \leqslant A \leqslant 1+c\delta\left(\begin{pmatrix}1\\ k-i\end{pmatrix}, \begin{pmatrix}1\\ k-i+1\end{pmatrix}\right) \\ \sim 1+\frac{c}{k-i} \end{aligned}$$

where c is a constant.

For the denominators, use that

$$\inf_j \frac{a_j}{b_j} \leqslant \frac{\sum_k a_k}{\sum_k b_k} \leqslant \sup_j \frac{a_j}{b_j}$$

to get similar bounds and finally

$$\left(1-\frac{c}{k-i}\right)^2 \leqslant \frac{m_k(x_0 \cdots x_{i-1}\,|x_i|\,x_{i+1} \cdots x_k \cdots)}{m(x_0 \cdots x_{i-1}\,|x_i|\,x_{i+1} \cdots x_k \cdots)} \leqslant \left(1+\frac{c}{k-i}\right)^2$$

We also have to bound the quotient

$$\begin{aligned} &\frac{dm_k(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)}{dm(x_0 \cdots x_{i-1}, x_{i+1} \cdots x_k \cdots)} \\ &\quad = \Bigg[\sum_{x_i} |M(x_0) \cdots M(x_{i-1})\, M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \tilde{u}_{k+1}|^\beta \\ &\quad\quad \times \pi(x_0 \cdots x_k)\, \gamma^{F(x_0 + \cdots + x_k)}\Bigg] \\ &\quad\quad \times \Bigg[\sum_{x_{i_\bullet}} |M(x_0) \cdots M(x_{i-1})\, M(x_i)\, M(x_{i+1}) \cdots M(x_k)\, \hat{u}_{k+1}|^\beta \\ &\quad\quad \times \pi(x_0 \cdots x_k)\, \gamma^{F(x_0 + \cdots + x_k)}\Bigg]^{-1} \end{aligned}$$

This quotient R can be analogously bounded by $(1-c/(k-i)) \leqslant R \leqslant (1+c/(k-i))$.

Finally,

$$\int_{\Omega/x_i} dm \sum_{x_i} x_i m(\cdots |x_i| \cdots) + \int_{\Omega/x_i} dm \sum_{x_i} \left[\left(1 - \frac{c}{k-i}\right)^3 - 1 \right]$$

$$\leqslant \int x_i \, dm_k$$

$$\leqslant \int_{\Omega/x_i} dm \sum_{x_i} x_i m(\cdots |x_i| \cdots) + \int_{\Omega/x_i} dm \sum_{x_i} x_i \left[\left(1 + \frac{c}{k-i}\right)^3 - 1 \right]$$

But

$$\frac{1}{k} \int \sum_{i=0}^{k} \frac{x_i}{k-i} \, dm \to 0$$

(because

$$\frac{1}{k} \sum_{i=0}^{k-1} \frac{1}{k-i} \sim \frac{1}{k} \int_{1/k}^{1-1/k} \frac{dx}{1-x} \to 0 \qquad \text{if} \quad k \to \infty$$

and $\int x_i \, dm$ is a finite quantity). Similarly the other terms containing powers of $1/k-i$) go to zero if $k \to \infty$, and this proves the claim.

Recall that there exists an ergodic measure absolutely equivalent to m, $hm \equiv \tilde{\mu}^+$. Therefore, in order to prove that $\lambda_2(k) \to \lambda_2$ when $k \to \infty$, it suffices to show that

$$\frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} x_i h \, dm \qquad \text{and} \qquad \frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} x_i \, dm$$

have the same limit behavior.

To show this fact, it is enough to observe that by the ergodicity of hm, and since $x_i \in L^2(h \, dm)$, we have

$$\frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} x_i \, dm = \frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} (x_0 \circ \tau^i)(h^{-1}) h \, dm$$

$$\to \int x_0 h \, dm \int h^{-1} h \, dm = \int x_0 h \, dm$$

and

$$\frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} x_i h \, dm = \frac{1}{k} \int_\Omega \sum_{i=0}^{k-1} (x_0 \circ \tau^i) \, h \, dm \to \int x_0 h \, dm$$

3.8. Entropy

Let $P^{(0)}$ be the partition (by cylinders) $\{C^0_{x_0}\}_{x_0 \in N}$. We denote $P^{(0)} \equiv P$. Here $P^{(n)}$ is the partition whose atoms are the cylinders $\{C^{0 \cdots n}_{x_0 \cdots x_n}\}_{x_i \in N}$. The partition P is countable, it has finite entropy, and it is generating. By the Sinai theorem, if $n \to \infty$,

$$\frac{1}{n} H_{\tilde{\mu}^+}(P \vee T^{-1}P \vee \cdots \vee T^{-n}P) \to h(\tilde{\mu}^+)$$

where

$$H_\mu\left(\bigvee_0^n T^{-i}P\right) = H_\mu(P^{(n)}) = \sum_{A \in P^{(n)}} \mu(A) \log \mu(A)$$

Similarly, for the measure m_k we have

$$\frac{1}{k} H_{m_k}(P^{(k)}) = \frac{1}{k} H_{m_k}(P \vee T^{-1}P \vee \cdots \vee T^{-1}P)$$
$$= -\frac{1}{k} \sum_{A \in P^{(k)}} m_k(A) \log m_k(A)$$

Proposition 3.7. We have

$$\frac{1}{k} H_{m_k}(P^{(k)}) \to h(\tilde{\mu}^+) \qquad \text{as} \quad k \to \infty$$

Proof. Compare first H_{m_k} and H_{m}. Arguments analogous to those of Lemma 3.6 show that $(1/k) H_{m_k}(P^{(k)})$ has the same limit behavior as $(1/k) H_m(P^{(k)})$.

Let $P^{(k)}$ the partition by atoms $x_0 \cdots x_k$. We have

$$H_{m_k}(P^{(k)}) = -\frac{1}{k} \int_\Omega dm_k(\mathbf{x}) \log m_k(P^{(k)})$$
$$= -\frac{1}{k} \int_\Omega dm_k(\mathbf{x}) \log m_k(x_0 \cdots x_k)$$
$$-\frac{1}{k_*} \sum_{i=0}^{k} \int_\Omega dm_k(\mathbf{x}) \log m_k(x_i \mid x_{i+1} \cdots x_k)$$
$$-\frac{1}{k} \sum_{i=0}^{k} \int_{\Omega/x_i} dm_k(\tilde{x}_0 \cdots \tilde{x}_{i-1}, \tilde{x}_{i+1} \cdots \tilde{x}_k \cdots)$$
$$-\sum_{x_i} m_k(\tilde{x}_0 \cdots \tilde{x}_{i-1} \,|x_i|\, \tilde{x}_{i+1} \cdots \tilde{x}_k \cdots) \log m_k(x_i \mid \tilde{x}_{i+1} \cdots \tilde{x}_k)$$

Again, choose i and write

$$-\sum_{x_i} m(\cdots|\cdots)\left(1-\frac{c}{k-i}\right)\log m(\cdots|\cdots)\left(1-\frac{c}{k-i}\right)$$

$$\leqslant -\sum_{x_i} \frac{m_k(\cdots|\cdots)}{m(\cdots|\cdots)} m(\cdots|\cdots)\log \frac{m_k(\cdots|\cdots)}{m(\cdots|\cdots)} m(\cdots|\cdots)$$

$$\leqslant -\sum_{x_i} m(\cdots|\cdots)\left(1-\frac{c}{k-i}\right)\log m(\cdots|\cdots)\left(1-\frac{c}{k-i}\right)$$

Then

$$H_{m_k}(P^{(k)}) = -\frac{1}{k}\sum_{i=0}^{k}\int_\Omega dm(\mathbf{x})\log m\left(T^{-i}P^{(0)}\,\Big|\,\bigvee_{j=i}^{k} T^{-j}P^{(0)}\right)+r(k)$$

The first term goes to $h(m)$ and $r(k)$ goes to zero when $k\to\infty$.

We have only to show that $h(m)=h(hm)$, where $hm\equiv\tilde{\mu}^+$ is the ergodic measure equivalent to m. We have the following lemma.

Lemma 3.8. Let P be a finite entropy partition. Then there exists a constant C independent of P such that

$$|H_{hm}(P)-H_m(P)|\leqslant C$$

Proof. Let $d\mu = h\,dm$. Write

$$|H_m - H_\mu| = \left|\int d\mu \int \frac{dm\,d\mu}{d\mu}\log\frac{m(P(x))\,\mu(P(x))}{\mu(P(x))}\right.$$

$$\left.-\int dm\int\frac{d\mu\,dm}{dm}\log\frac{\mu(P(x))\,m(P(x))}{m(P(x))}\right|$$

$$=\left|\int d\mu\,dm\log\frac{m(P(x))}{\mu(P(x))}\right|\leqslant \log\|h\|_\infty$$

It follows that

$$\left|\frac{1}{k}H_m(P^{(k)})-\frac{1}{k}H_{hm}(P^{(k)})\right|\leqslant\frac{C}{k}$$

so we conclude that $(1/k)\,H_{m_k}(P^{(k)})$, which has the same limit behavior as $(1/k)\,H_m(P^{(k)})$, also has the same limit behavior as $(1/k)\,H_{hm}(P^{(k)})$, which goes to $h(\tilde{\mu}^+)$.

3.9. The Conditional Entropy h_1

Let $P^{(n)}$ be the partition by cylinders $C^{0\cdots n}_{x_0\cdots x_n}$. Let P_n be the partition of the square $[0, 1] \times [0, 1]$ corresponding to $P^{(n)}$. Let Q_n be the "vertical" partition $P_n \times [0, 1]$. Define the conditional entropy of m_k, $h_1(m_k)$, by

$$h_1(m_k) = -\frac{1}{k}\int dm_k(\mathbf{x}) \log\left(\frac{m_k(P_k \cap Q_k)}{m_k(Q_k)}\right)$$

$$= -\frac{1}{k}\int_\Omega dm_k(x_0 \cdots x_k \cdots) \log |M(x_0) \cdots M(x_k) u_{k+1}|$$

We show that $h_1(m_k)$ has a limit when $k \to \infty$, which we denote "lim," and that this "lim" is greater than or equal to the conditional entropy of $\tilde{\mu}^+$, $h_1(\tilde{\mu}^+)$.

More precisely, we show that $h_1(m_k)$ has the same limit, for $k \to \infty$, as $H_{\tilde{\mu}^+}(P_k \mid Q_k)$, this limit being an upper bound for $h_1(\tilde{\mu}^+)$.

Lemma 3.9. We have

$$\lim_{n\to\infty} H_{\tilde{\mu}^+}(P_n \mid Q_n) = \lim_{n\to\infty} h_1(m_n)$$

Proof. This is again the same argument. Write

$$h_{m_k}(P_n \mid Q_n) = \frac{1}{k}\int dm_k(\mathbf{x}) \log |M(x_0) \cdots M(x_k) u_k|$$

$$= \frac{1}{k}\sum_{i=0}^{k}\int dm_k(\mathbf{x}) \log \frac{|M(x_i)\, u_{i+1}|}{|u_{i+1}|}$$

where $u_{i+1} = M(x_{i+1}) \cdots M(x_k) u_k$, and $|u_i| = 1$.

By the same arguments, this quantity has the same limit behavior as

$$\frac{1}{k}\sum_{i=0}^{k}\int dm(\mathbf{x}) \log |M(x_i) u_{i+1}|$$

$$= \frac{1}{k}\sum_{i=0}^{k}\int dm(\mathbf{x})\, F_k(\tau^i \mathbf{x}) \to \int d\tilde{\mu}^+ F(\mathbf{x}) = \text{"lim"}$$

where $F(\mathbf{x}) = \log |M(x_0) u_1|$.

We have the following proposition (m being the weak limit of the m_n).

Proposition 3.10. We have

$$h_1(\tilde{\mu}^+) \leqslant \lim h_1(m_k)$$

Proof. Let P_n be fixed; there exists a vertical partition $\tilde{Q}_m(n)$, with $|\tilde{Q}_{m(n)}| \to 0$ when $n \to \infty$, such that the limit h_1 is attained for the partition $P_n \mid \tilde{Q}_{m(n)}$:

$$h_1(\tilde{\mu}^+) = \lim_{n \to \infty} \log \tilde{\mu}^{+ss}(P_n) = \lim_{n \to \infty} -\frac{1}{n} H_{\tilde{\mu}^+}(P_n \mid \tilde{Q}_{m(n)})$$

where $\tilde{\mu}^{ss}$ is the vertical conditional measure of $\tilde{\mu}^+$. We have that $\tilde{Q}_{m(n)} \cap P_n \subseteq P_n$. Then $H_{\tilde{\mu}^+}(P_n \mid \tilde{Q}_{m(n)}) \leqslant H_{\tilde{\mu}^+}(P_n \mid Q_n)$ because $\tilde{Q}_{m(n)}$ raffines Q_n, Q_n being the partition $P_n \times [0, 1]$.

Now,

$$\lim_{n \to \infty} H_{\tilde{\mu}^+}(P_n \mid Q_n) = \lim_{n \to \infty} h_1(m_n)$$

It follows that

$$\begin{aligned} h_1(\tilde{\mu}^+) &= \lim_{n \to \infty} H_{\tilde{\mu}^+}(P_n \mid \tilde{Q}_{m(n)}) \\ &\leqslant \lim_{n \to \infty} H_{\tilde{\mu}^+}(P_n \mid Q_n) = \lim_{n \to \infty} h_1(m_n) = \text{“lim”} \end{aligned}$$

4. CONCLUSION

Set

$$\begin{aligned} G_k = \sum_{x_1 \cdots x_n} & \left(\frac{M(x_0)\, M(x_1) \cdots M(x_n)(v_{n+1})}{2^{x_0 + x_1 \cdots x_n}} \right)^{\beta} \\ & \times \gamma^{F(x_0 + x_1 + \cdots + x_n)}\, \pi(x_0 x_1 \cdots x_n) \end{aligned}$$

A simple identity between the partial derivatives of G_k yields

$$h_2(m_k) - h_1(m_k) = \lambda_2(m_k)(-f_k(\alpha_k + \delta) - (\alpha_k + \delta)) \qquad (*)$$

By Propositions 3.7 and 3.10 the limit of the l.h.s. of $(*)$ satisfies

$$h_2(\tilde{\mu}^+) - \text{“lim”} \leqslant h_2(\tilde{\mu}^+) - h_1(\tilde{\mu}^+)$$

On the other hand, by Lemma 3.6 and the continuity of f

$$\lim_{k \to \infty} \lambda_2(m_k)[-f_k(\alpha_k + \delta) - (\alpha_k + \delta)] = \lambda_2(\tilde{\mu}^+)[f(\alpha + \delta) - (\alpha + \delta)]$$

Finally, by ref. 14 we know that the transverse dynamics relative to the two-dimensional system $(\Omega, \tilde{\mu}^+, \tau)$ obeys the relation

$$h_2(\tilde{\mu}^+) - h_1(\tilde{\mu}^+) = \lambda_2(\tilde{\mu}^+)\, \gamma(\tilde{\mu}^+) \qquad \text{(LY)}$$

where $\gamma(\tilde{\mu}^+)$ is the dimension of the projected measure of $\tilde{\mu}^+$. Then, by combining these relations, we have

$$\lambda_2(\tilde{\mu}^+)\,\gamma(\tilde{\mu}^+) = h_2(\tilde{\mu}^+) - h_1(\tilde{\mu}^+) \geqslant h_2(\tilde{\mu}^+) - \text{“lim”}$$
$$= \lambda_2(\tilde{\mu}^+)[f(\alpha+\delta) - (\alpha+\delta)]$$

This means

$$\dim \nu^* \geqslant -f(\alpha+\delta) - (\alpha+\delta)$$

since ν^* is equivalent to the projected measure of $\hat{\mu}^+$.

On the other hand (cf. Section 3.6), ν^* is supported by $pS(\alpha+\delta, \alpha_2)$. This shows that actually $HDpS(\alpha+\delta, \alpha_2) \geqslant -f(\alpha+\delta) + (\alpha+\delta)$.

ACKNOWLEDGMENT

It is our pleasure to thank warmly Pierre Collet.

REFERENCES

1. J. Alexander and J. Yorke, The fat baker's transformation, *Ergod. Theory Dynam. Syst.* **4**:1–23 (1984).
2. J. Alexander and D. Zagier, The entropy of a certain infinitely convolved Bernoulli measure, University of Maryland preprint (1991).
3. P. Billingsley, *Probability and Measure* (Wiley, New York, 1975).
4. Ph. Bourgerol, Théorèmes limites pour les systèmes linéaires à coefficients markoviens, *Prob. Theory Related Fields* **78**:193–221 (1988).
5. Ph. Bourgerol and J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators* (Birkhäuser, Basel, 1985).
6. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (Springer, New York, 1975).
7. E. Cohen, H. Kesten, and C. Newman, *Random Matrices and Their Applications* (American Mathematical Society, Providence, Rhode Island, 1984).
8. P. Collet, J. Lebowitz, and A. Porzio, The dimension spectrum of some dynamical systems, *J. Stat. Phys.* **47**(5/6):609–644 (1987).
9. M. Denker, C. Grillenberger, and K. Sigmund, *Ergodic Theory on Compact Spaces* (Springer-Verlag, Berlin, 1976).
10. Y. Guivarc'h, Exposants caractéristiques des produits de matrices aléatoires en dépendence markovienne, in *Probability Measures on Groups* (Springer-Verlag, Berlin, 1984).
11. T. Halsey, M. Jensen, L. Kadanoff, I. Procaccia, and B. Shraiman, Fractal measures and their singularities: The characterisation of strange sets, *Phys. Rev. A* **33**:1141–1151 (1986).
12. O. Lanford, Entropy and equilibrium states in classical statistical mechanics, in *Lecture Notes in Physics*, No. 20 (Springer, Berlin, 1971).
13. F. Ledrappier and A. Porzio, A dimension formula for Bernoulli convolutions, *J. Stat. Phys.* **76**(5/6):1307 (1994).
14. F. Ledrappier and L. S. Young, The metric entropy of diffeomorphisms, *Ann. Math.* **122**:540–574 (1985).

15. K. R. Parthasaraty, *Probability Measures on Metric Spaces* (Academic Press, New York, 1967).
16. D. Placky and J. Steinebach, *Period. Math.* **6**:338–340 (1974).
17. D. Ruelle, Thermodynamic formalism, in *Encyclopedia of Mathemtics* (Addison-Wesley, Reading, Massachusetts, 1978).
18. Ya. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surv.* pp. 21–69 (1975).